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Electromagnetic Relations in a Single Coordinate System*

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Certain experimental phenomena, like the homopolar generator, whose analyses have caused confusion are shown to be understandably treated by the following: Use the differential form of Maxwell's four field relations together with the Lorentz force equation as the fundamental relations for treating all electromagnetic phenomena in the observer's coordinate system. The Lorentz force is considered to act on all of the charges in any material body moving in magnetic fields in these coordinates. Faraday's law $\mathcal{E}=-d\Phi/dt$, becomes a derived through very useful relation. This system provides revealing Poynting vector for all steady-state energy conversions.

While the concepts concerning electromagnetism as covered by Maxwell's equations have been well established for some time, concerning certain phenomena, there still exists some confusion in the minds of many who should have mastered the subject. This fact has been impressed upon me by discussions with colleagues and with graduate and undergraduate students. For example, the analysis of emf's produced by homopolar generators, Faraday disks, and similar devices still cause considerable difficulty. That others have noticed this confusion is attested by the number of articles that have been published recently on this type of experiment. At least three[1–3] have appeared in this journal since June 1962. In analyzing these experiments each author has used a different procedure through all necessarily reached the same conclusion.

A certain rigorous analytical procedure for analyzing macroscopic[4] experiments in electromagnetism, which is also simple enough for undergraduate science students to understand, seems to have been overlooked. This procedure has the distinct advantage that all symbols refer to quantities measured in the observer's own coordinate system. Of course, it must be and is possible to use this procedure for analyzing a given phenomenon in either of two systems moving at constant velocity with respect to each other. The ability to do this provides students with an insight into relativity principles. The procedure not only predicts correctly the results of experiments but also provides a Poynting-vector sink wherever electromagnetic energy is being converted to some other form of energy and a Poynting-vector source wherever the conversion is progressing in the opposite direction. This facilitates the analyses of power conversions. It is far from obvious that the other procedures share this last advantage especially in spaces where fields do not vary with time. This procedure is not new and differs very little from those set forth in most texts.

It merely uses the fact that all macroscopic electromagnetic phenomena are included in the following differential form of Maxwell's four field equations

$$\nabla\cdot\mathbf{D}=\rho,\qquad \nabla\cdot\mathbf{B}=0,$$
$$\nabla\times\mathbf{E}=-\dot{\mathbf{B}},\quad \nabla\times\mathbf{H}=\mathbf{J}+\dot{\mathbf{D}},$$

together with the Lorentz force equation

$$\mathbf{F}=q(\mathbf{E}+\mathbf{v}\times\mathbf{B}),$$

where the dot over any symbol denotes partial differentiation with respect to time. The force equation is needed for defining the field vectors **E** and **B** and for analyses of results. Since **D** and **H** also appear in the four equations, the defining equations $\mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P}$ and $\mathbf{B}=\mu_0\mathbf{H}+\mathfrak{M}$ must also appear.[5] While the law of conservation of electric

* Actually, the single coordinate system must be an inertial system. For practical purposes coordinates fixed to the earth can be considered inertial.

[1] J. W. Then, Am. J. Phys. **30**, 411 (1962).

[2] A. K. Das Gupta, Am. J. Phys. **31**, 428 (1963).

[3] D. L. Webster, Am. J. Phys. **31**, 590 (1963).

[4] Though the relations adopted here refer specifically to macroscopic quantities, they also apply to microscopic and submicroscopic phenomena. In the latter case, analysis also requires the introduction of wave-mechanical concepts.

[5] In the notation used here **E** represents the Coulomb fields due to all charges plus the nonconservative fields induced by varying magnetic fields, and **B** represents the magnetic fields due to all charge motions. Since **P** and **M** are necessarily obtained from averages over small macroscopic volumes, **D** and **H** are also volume averages. This means that similar volume averages must be used for **E**

charge is one of the fundamental laws of nature, it is not necessary to include it as a separate relation. It is included in Maxwell's equations and is easily obtained from a simple mathematical combination of the first and fourth field relations.

Likewise, it is not necessary to include Faraday's law for electromagnetic induction, since it is obtainable directly from Maxwell's third relation, and the Lorentz force equation. In fact, the introduction of this law, $\mathcal{E}=-d\Phi/dt$, does cause considerable confusion in the minds of many scientists and especially in the minds of beginning students. While Faraday's experiments were important to the foundation of Maxwell's equations and the Lorentz law, there is no more reason to consider the Faraday law as fundamental than to so consider Lenz's law. Many of the best texts on this subject assume the historical point of view taken by Maxwell, that Faraday's law is fundamental and can be used to derive the third differential equation for the special case where there is no motion or $\mathbf{v}=0$ in the observer's coordinate system. Actually the differential form of the third relation given here is generally valid without this restriction. Application of Stoke's theorem yields the form $\oint\mathbf{E}\cdot d\mathbf{l}=-\int\dot{\mathbf{B}}\cdot d\mathbf{s}$. Adding $\oint\mathbf{v}\times\mathbf{B}\cdot d\mathbf{l}$ from the Lorentz force relation to both sides of this equation gives the induced emf $\mathcal{E}$ on the left and $-(d/dt)\int\mathbf{B}\cdot d\mathbf{s}=-d\Phi/dt$ on the right. Thus Faraday's law becomes a derived relation that is very useful.

In the opening paragraph of his "Experimental Study of the Motional Electromotive Force" John W. Then[1] states that most of his experiments were designed to eliminate "from the circuit ··· a change in flux $d\Phi/dt$ threading the circuit" and "a change in intensity dB/dt of the magnetic field." He then proceeds to show experimentally that, even when these two conditions are satisfied, steady rotations of conductors in magnetic fields do produce steady emf's. When asked to predict the results of such experiments in advance, students who have learned to analyze problems only in terms of Faraday's law, will invariably reach incorrect conclusions.

Correct analyses for Then's experiments are obtainable directly from the Lorentz force equation. In each experiment where the galvanometer showed induced currents, conductors were moving with various velocities through constant magnetic fields. All of the charges in these conductors experience forces proportional to $\mathbf{v}\times\mathbf{B}$ but only the current carriers (the conduction electrons) are free to drift through the lattice. The emf induced in the circuit containing the moving couductors is given by

$$\mathcal{E}_m=\oint\mathbf{v}\times\mathbf{B}\cdot d\mathbf{l}=-\oint\mathbf{B}\cdot\mathbf{v}\times d\mathbf{l}.$$

If $\mathbf{B}$ changes with time an additional emf is obtained from $\mathcal{E}_t=-\int_s\dot{\mathbf{B}}\cdot d\mathbf{s}$, integrated over a surface whose periphery coincides with the closed path of the line integral. This last integral is obtained by integrating Maxwell's third equation over such a surface and transforming the left side of a line integral by Stoke's theorem. The general expression for magnetically induced emf's then can be written

$$\mathcal{E}=\mathcal{E}_t+\mathcal{E}_m=-\int_s\dot{\mathbf{B}}\cdot d\mathbf{s}-\oint\mathbf{B}\cdot\mathbf{v}\times d\mathbf{s}. \quad (1)$$

Here $\mathbf{v}$ is the velocity in the observer's coordinate system of the moving conductor at the line element $d\mathbf{l}$. Corson[6] gives this derivation and also shows that the expression for the induced emf given here can be obtained directly from Faraday's law. There are probably two reasons why basing these analyses on $\mathcal{E}=-d\Phi/dt$ causes confusion: (1) a rather sophisticated differentiation, under an integral sign, together with a careful definition of symbols, is required to obtain the second or $\mathcal{E}_m$ term and (2) the law is only valid in its differential form, since direct integration produces the nonsensical result $\Phi=-\mathcal{E}_m t+C$ for the flux through any of Then's circuits. Many authors resort to transformations to other co-

and **B**. The volume average over a sphere of the Coulomb field due to an elementary charged particle at the center of that sphere is zero. Furthermore, for charges at rest $\oint\mathbf{E}\cdot d\mathbf{l}=0$ for any path. The volume average of **E** is not affected by the neutral atoms of the crystal. In using the Lorentz force equation to define **E** and **B** any "forces" arising from quantum mechanical effects must be eliminated. Under these restrictions the five basic equations listed here can be made valid in submicroscopic regions by setting $\mathbf{P}=\mathbf{M}=0$.

[6] D. R. Corson, Electromagnetic Induction in Moving Systems; Am. J. Phys. **24**, 126 (1956).

ordinate systems for analyzing such experiments. The advantage of the second term in Eq. (1) is that the $\mathbf{v}\times\mathbf{B}$ in it focuses attention on the need for motion of some material media relative to the observer's coordinate system in a magnetic field. The use of Faraday's law tends to focus attention on change of flux, which is difficult to visualize in experiments similar to Then's. It should be noticed that while the first term of Eq. (1) provides emf's even in a vacuum (e.g., in Kersts' betatron), some material medium must move in a magnetic field to produce an observable emf due to motion.

For Ohmic conductors at rest in the laboratory most texts show that $\mathbf{E}=\mathbf{J}\rho_R$ where ρ_R is the resistivity and $\mathbf{J}$ the current density. The conduction electrons in conductors moving in a constant magnetic field also experience the force $\mathbf{v}\times\mathbf{B}$. Thus the expression becomes

$$\rho_R\mathbf{J}=\mathbf{E}+\mathbf{v}\times\mathbf{B}. \qquad (2)$$

This expression[7] is very useful for analyzing linear generators, homopolar generators and motors, and ordinary dc generators and motors. For simplicity assume $\mathbf{B}$ independent of time and ignore field windings. By Maxwell's third equation $\nabla\times\mathbf{E}=0$ so that $\mathbf{E}$ is conservative and $\oint\mathbf{E}\cdot d\mathbf{l}=0$ for any path. The $\mathbf{v}\times\mathbf{B}$ term is responsible for the steady current. It is possible to connect any one of the machines to some storage battery and drive it at such a speed that there is no current through the machine. Then $\mathbf{J}=0$ and $\mathbf{E}=-\mathbf{v}\times\mathbf{B}$. The magnitude of $\mathbf{v}\times\mathbf{B}$ can be increased by increasing $\mathbf{v}$ to charge the battery or can be decreased by decreasing $\mathbf{v}$ to make the battery run the machine. In both cases conversion of power takes place in the machine. Mechanical power must be fed into the machine when it is charging the battery and mechanical power is drawn from the machine when the battery runs it as a motor.[8] Let $v=v_0$ when $\mathbf{E}+\mathbf{v}\times\mathbf{B}=0$. When $v>v_0$ both sides of $\rho_R\mathbf{J}=\mathbf{E}+\mathbf{v}\times\mathbf{B}$ are directed opposite to $\mathbf{E}$ to make $\mathbf{E}\cdot\mathbf{J}$ negative but when $v<v_0$ the dot product $\mathbf{E}\cdot\mathbf{J}$ is positive. When $\mathbf{E}\cdot\mathbf{J}$ is negative as in the first case mechanical power is being converted to electrical power, whereas, when $\mathbf{E}\cdot\mathbf{J}$ is positive as in the second case the direction of conversion is reversed. The Joulean heat $J^2\rho_R$ is lost in both cases.

The Poynting vector $\mathbf{S}=\mathbf{E}\times\mathbf{H}$ provides simple and revealing analyses in such cases.[9] Directly from Maxwell's equations

$$\nabla\cdot\mathbf{S}+\mathbf{E}\cdot\dot{\mathbf{D}}+\mathbf{H}\cdot\dot{\mathbf{B}}+\mathbf{E}\cdot\mathbf{J}=0$$

or where $\dot{\mathbf{D}}=\dot{\mathbf{B}}=0$ as in dc networks,

$$\nabla\cdot\mathbf{S}=-\mathbf{E}\cdot\mathbf{J}. \qquad (3)$$

For any imaginary closed surface s drawn about one of these machines

$$\int_s\mathbf{S}\cdot d\mathbf{s}=-\int_\tau\mathbf{E}\cdot\mathbf{J}d\tau$$

where τ represents the volume inside s. The left side gives the electromagnetic power emerging from the machine at any instant. For this to be positive $\mathbf{E}\cdot\mathbf{J}$ must be predominantly negative. From Eq. (2)

$$\mathbf{E}=\rho_R\mathbf{J}-\mathbf{v}\times\mathbf{B}$$

and

$$\mathbf{E}\cdot\mathbf{J}=\rho_RJ^2-\mathbf{v}\times\mathbf{B}\cdot\mathbf{J}$$

Now $\int_\tau\rho_RJ^2d\tau$ is the Joulean heat power lost in the machine, which is quite small and can be neglected in well-designed machines. When the machine acts as a generator $\mathbf{J}$ is nearly parallel to $\mathbf{v}\times\mathbf{B}$ so that $\mathbf{E}\cdot\mathbf{J}$ *is* negative and electromagnetic power does emerge from the machine. On the other hand, when any of these machines

[7] When a wire moves in a magnetic field $\mathbf{v}\times\mathbf{B}$ may not be parallel to the axis of the wire. In such a case electric charges collect on the surface of the wire to prevent current from going out into space, thus $\mathbf{E}+\mathbf{v}\times\mathbf{B}$ is always parallel to the axis of such a wire. Here $\mathbf{v}$ is the velocity of the element of the conductor that is being considered. Though the average velocity of the conduction electrons is $\mathbf{v}_a=\mathbf{v}+\mathbf{u}$, where $\mathbf{u}$ is the drift velocity relative to the conductor, $\mathbf{u}\times\mathbf{B}$ has no component parallel to $\mathbf{J}$ or the axis of the wire and can be neglected here. The force containing $\mathbf{u}\times\mathbf{B}$ does account for the mechanical power being converted by the machine.

[8] In machines acting as dc generators, the mechanical power is usually delivered to the rotors through metallic axles and metallic frames, which consist of neutral atoms formed into stable crystals. Coulomb forces between the charged particles of these atoms play a large role in the mechanical strength of these crystals. However, the equilibrium displayed by these crystals could not be achieved by Coulomb forces alone as is demonstrated by Earnshaw's theorem. Quantum mechanical principles must be invoked to achieve this equilibrium.

[9] Pugh and Pugh, *Principles of Electricity and Magnetism* (Addison-Wesley Publishing Company, Reading, Massachusetts (1960), p. 379.

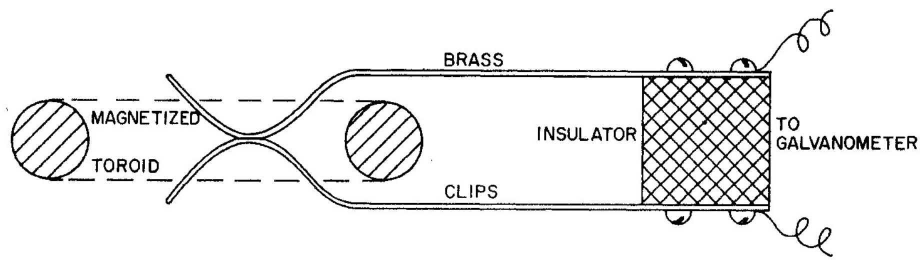

FIG. 1. Schematic of toroid.

operates as a motor **J** is directed nearly opposite to **v** × **B**, which makes **E**·**J** positive and electromagnetic power is flowing into the machine. For this analysis **v** × **B** should *not* be considered to be an electric field. If it is designated as an electric field **E**′, then **E**′ must not be included with **E** in the Poynting-vector analysis. I find it less confusing if **E**′ is never introduced. Since the Poynting vector is defined only in terms of its divergence, functions whose divergences are zero throughout space should be ignored.

The logical extension of this analysis is to presume that electromagnetic energy flows out from every source of dc emf that is delivering current to external circuits. This demands that **E**·**J** be predominantly negative within such sources. In a recent paper[10] I showed that according to Maxwell's third equation **E** is always conservative in dc networks and hence **J** and **E** must be oppositely directed in all sources of dc emf when they are delivering power to external networks. On the other hand, within a storage battery being charged, for example, **J** and **E** are parallel, **E**·**J** is positive and electromagnetic power is being absorbed and converted to chemical energy. If, as has often been stated, **J** and **E** must always have the same direction in dc networks we would be faced with *an anomalous situation. Throughout such a space we should have many sinks for the Poynting vector but no sources.*

To illustrate the usefulness of the procedures outlined here, there are two easily performed experiments, which have received little if any publicity: (1) Consider two large metallic spheres separated from each other and electrically isolated from their surroundings but connected to each other through a high resistance. By some technique (like the use of an electrophorus) students can pick up negative charges from one sphere and carry these to the other. With sufficiently high resistance and with a large number of students, a nearly steady potential difference can be maintained between the spheres to produce a nearly steady current through the resistance. The students constitute the source of emf. They do work carrying charges against the electrostatic field. According to some analytical procedures these students would have to be designated as consituting an electric field **E**′. In the procedure proposed here it is merely stated that the students are converting some other form of energy to the electromagnetic form. (2) Consider a toroid made of homogeneous and isotropic ferromagnetic metal (Fig. 1). If this is permanently magnetized parallel to the circle passing through the centers of its circular cross sections, no magnetic field exists outside the metal. Provide two spring brass clips fastened to an insulating block as shown in the diagram. With the two clips connected to a ballistic galvanometer, the clips can be slid back and forth across the toroid without breaking the galvanometer circuit. In the position shown in the diagram all of the magnetic flux Φ_m passing through the toroid passes through the galvanometer circuit. After this device is pulled to the right over this toroid no magnetic flux threads the circuit. The flux through the circuit has been changed from Φ_m to zero without breaking the circuit. Does the galvonometer receive a ballistic impulse? Since no part of the circuit moves

[10] E. M. Pugh, Conservative Fields in dc Networks; Am. J. Phys. p. 29, 484, (1961). This and the present paper are complementary and should be read together.

in a magnetic field, Eq. (1) predicts a null result. This prediction is correct. The galvanometer does not deflect.[12] A perfect toroid is difficult to construct but a magnetron magnet having its gap filled with a soft iron cylinder can be used instead. Since in this case the magnetic fields surrounding the iron cylinder are not quite zero, some deflection will be observed but it will be much smaller than when the brass clips are pulled through the gap with the iron cylinder removed.

[12] Instead of moving the brass clips to the right over the toroid, the clips can be held still and the toroid moved to the left. In that case the metallic toroid moves in its own magnetic field. The second term of Eq. (1) is not zero but neither is the first. These two terms are of opposite sign and exactly cancel so that $\mathcal{E}=0$ in this case also. The two cases are thus analyzed consistently in a single coordinate system but the results are in agreement with the relativity principle. This helps make the meaning of the relativity principle clearer to students.

CONCLUSION

It is shown that certain concepts in electromagnetic theory that still cause confusion can be simplified by analyzing them in a single coordinate system solely on the basis of the differential form of Maxwell's four field equations together with the Lorentz force equation. Neither the conservation of charge nor Faraday's law of induction need be introduced separately. The procedure outlined provides easily visualized expressions for analyzing all rotating machines, including the homopolar types. In particular, it provides consistent analyses of all power conversions by means of the Poynting vector.

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On Certain "Ghost" Lines Observable with a Prism Spectrometer, and Their Use in Precision Refractometry

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When a prism with three polished faces is used on a spectrometer, because of partial internal reflection, light eventually emerges from all three faces of the prism. It is shown that, in general, there are only two directions of emergence from each face. From the face of incidence one of these directions is uniquely that of the externally reflected light. The two emergent beams from any face are unequally dispersed, and in certain circumstances one may be undispersed (one of the beams from the face of incidence is always of this character). In such circumstances the undispersed beam coincides in direction with each component of the dispersed beam when that component is at minimum deviation. This property provides an accurate method of adjustment for minimum deviation. When this method is used in practice, there are important advantages to be gained by heavily silvering two of the prism faces and making observations on the spectrum observed in a telescope directed towards the unsilvered face of incidence.

INTRODUCTION

IN the course of a routine experiment with a prism spectrometer, using a mercury-vapor light source, one of us (JKM) observed a single undispersed "ghost" line, of intensity comparable with that of the blue ($\lambda=4916$ Å) line of the mercury spectrum. This ghost line was visible in the same field of view as the normal spectrum only when the adjustment of the instrument was close to that of minimum deviation, and, in this situation, as the prism was slowly rotated, the ghost line moved rapidly across the field. It soon became evident that the ghost line coincided with each normal spectrum line, in turn, as that line was itself at minimum deviation. These observations were made using a nominally equiangular prism of crown glass, having all three prism faces polished. The intensity of the ghost line was found to vary with the degree of internal reflection possible at the "unused" face

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