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Electromagnetic Induction in Moving Systems

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The principles underlying the extension of the Maxwell equation $\mathrm{curl}\,\mathbf{E} = -\partial\mathbf{B}/\partial t$ to moving media are discussed, and illustrations with specific moving systems are presented. The principles are applied to the case of the unipolar generator.

OUTLINE OF PRINCIPLES

THE subject of induced electromotive forces in moving conductors frequently confuses students, and in many textbooks the discussion is insufficient to clarify the matter. The application of the Maxwell equation

$$\mathrm{curl}\,\mathbf{E} = -\partial\mathbf{B}/\partial t \qquad (1)$$

to problems involving moving conductors, e.g., the unipolar generator, is especially confusing. The purpose of this paper is to state the principles clearly and then to illustrate them with specific examples.

The basis for the whole subject of induced electromotive forces is the Faraday induction law which states that

$$\oint \mathbf{E}\cdot d\mathbf{s} = -d\phi/dt, \qquad (2)$$

where the flux ϕ is given by

$$\phi = \int_S \mathbf{B}\cdot d\mathbf{S}. \qquad (3)$$

In a generalized statement of the law the path around which the line integral of the induced electric field intensity $\mathbf{E}$ is to be evaluated in Eq. (2) may be any closed path. It need not be confined to conducting material. The surface over which the flux is to be calculated in Eq. (3) may be any surface whose boundary is the integration path of Eq. (2).

Equation (2) is true, so far as experiment has been able to determine, no matter what the origin of the changing flux may be: the current producing the flux may change with time, the flux-producing circuit may move relative to the closed path around which the emf is being calculated, or the path, or parts of it, may move relative to the flux-producing circuit.

The Faraday law may be stated in differential form with the aid of Stokes' theorem:

$$\oint \mathbf{E}\cdot d\mathbf{s} = \int_S \mathrm{curl}\,\mathbf{E}\cdot d\mathbf{S} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}, \qquad (4)$$

where the surface over which $\mathrm{curl}\,\mathbf{E}$ is to be integrated may again be any surface bounded by the closed path of the line integral. In the special case where the path of integration in the line integral and the surface of integration in the integral are fixed in space we may interchange the order of differentiation and integration in the flux term in Eq. (4), so that

$$\int_S \mathrm{curl}\,\mathbf{E}\cdot d\mathbf{S} = -\int_S \frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S}. \qquad (5)$$

Here the partial derivative $\partial\mathbf{B}/\partial t$ has been used because the time derivative of $\mathbf{B}$ is to be evaluated at points on the fixed surface S. Equation (5) is valid for any surface, no matter how small, in the field so the integrands must be equal at every point. Thus

$$\mathrm{curl}\,\mathbf{E} = -\partial\mathbf{B}/\partial t. \qquad (6)$$

This is the Maxwell equation. The restriction of fixed path implicit in its derivation means that $\mathbf{E}$ and $\mathbf{B}$ must be measured in the same coordinate system. We will examine the meaning of this in the examples.

Now, what happens to Eq. (6) if the path and surface of integration are allowed to move in the field? We must go back to the Faraday law, which states that

$$\oint \mathbf{E}\cdot d\mathbf{s} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}. \qquad (7)$$

We must calculate the rate of change of flux through a surface bounded by a path which is allowed to move in an arbitrary manner. We can do this readily by applying the divergence theorem to the volume swept out by the surface in a time dt. The result is

$$\frac{d}{dt}\int_s \mathbf{B} \cdot d\mathbf{S} = \int_s \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} - \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{s}, \quad (8)$$

where $\mathbf{v}$ is the velocity of an element of the closed path around which the emf is to be calculated. The two terms of Eq. (8) represent (a) the change of flux by virtue of the change of $\mathbf{B}$ with time and (b) the flux lost (or gained) through the sides of the volume traced out by the moving path. For a general vector field there would be a third term involving the divergence of the vector, but in this regard we live in a simple world since div $\mathbf{B} = 0$ everywhere.

We may write Eq. (8) in differential form, as before, by using Stokes' theorem and setting the integrands equal so that at every point in the field

$$\mathrm{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} + \mathrm{curl}(\mathbf{v} \times \mathbf{E}). \quad (9)$$

In this equation $\mathbf{E}$ is the induced electric field intensity measured in a coordinate system which is moving with velocity $\mathbf{v}$ relative to the coordinate system in which the magnetic induction vector is measured to be $\mathbf{B}$. In practical cases where we are interested in the electric field induced in moving conductors, $\mathbf{v}$ is the velocity of the conductor relative to the laboratory and $\mathbf{B}$ is measured in the laboratory.

When we say that $\mathbf{E}$ is to be measured in a moving coordinate system, we mean that it is the field which would be experienced by a charge at rest in that system.

Example No. 1

To see what Eqs. (6) and (9) mean, consider the following example. Suppose we establish a uniform, time-independent, magnetic field in the laboratory and suppose we have a disk (supposed nonconducting and nonmagnetic for the moment) rotating with angular velocity ω about an axis perpendicular to its plane and parallel to $\mathbf{B}$, as in Fig. 1. Now suppose we

Fig. 1. A flat disk rotates with angular velocity ω about an axis perpendicular to its plane. The axis is parallel to the uniform magnetic field $\mathbf{B}$.
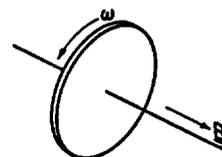
station two observers: one in the laboratory and one on the rotating disk. We equip each observer with a "curl meter," a "B-meter," and a stopwatch. The "curl meter" consists of a small loop of wire, capable of orientation in any direction, in series with a sensitive, infinite-impedance voltmeter.

Since

$$\mathrm{curl}_n\mathbf{E} = \lim_{S \to 0} \frac{\oint \mathbf{E} \cdot d\mathbf{s}}{S} \quad (10)$$

for the component of the curl normal to the plane containing the path of integration, the voltmeter reading divided by the area of the loop is the component of the curl in the direction chosen, if the loop is small enough. The "B-meter," ideally, can be an element of current-carrying conductor on which magnetic force can be measured. Or better, it can be a cathode-ray tube in which the deflection of the beam on the tube face is calibrated in webers/m². The stopwatch is used to measure the rate of change of $\mathbf{B}$.

Each observer knows nothing about the field except what he measures with his instruments.

What reading does each observer see on his instruments? The laboratory observer measures the uniform $\mathbf{B}$ we have established. He determines its direction and magnitude by observing the deflection on his cathode-ray beam for at least two mutually perpendicular orientations. He also observes that for each orientation the beam deflection is time-independent, so $\partial \mathbf{B}/\partial t = 0$. Furthermore his "curl meter" reads zero for all orientations since there is no changing flux through the loop. For the laboratory observer curl $\mathbf{E} = -\partial \mathbf{B}/\partial t = 0$.

When the disk observer points his electron beam parallel to the axis of his disk, he records no deflection so there is no component of $\mathbf{B}$ in the plane of the disk. When he points it anywhere in the plane of the disk, he records the same deflection, no matter where he is on the disk, so he concludes that the B-vector is uniform and perpendicular to his disk. He finds no
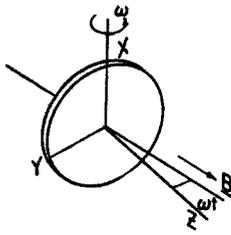
Fig. 2. A flat disk rotates with angular velocity ω about an axis in its plane. The axis of rotation is perpendicular to the uniform magnetic field **B**.

time dependence. In fact the field he measures is identical with the one which the laboratory observer measures. The "curl meter" on the disk sees only a constant flux so curl $\mathbf{E}=0$ everywhere on the disk.

The disk observer finds curl $\mathbf{E}=-\partial\mathbf{B}/\partial t=0$.

Example No. 2

Suppose now that we place a second disk in the field and rotate this one, with angular velocity ω, about an axis in the plane of the disk and perpendicular to the direction of **B**, as indicated in Fig. 2. An observer on this disk, equipped with the same instruments as the other observers, will ascribe entirely different properties to the field. Suppose that the disk observer establishes a coordinate system with the z-axis perpendicular to the plane of the disk and the x-axis parallel to the axis of the rotation (the disk observer knows nothing about an axis of rotation and it makes no difference how he chooses his axes, but our discussion is simpler if he does it this way). Suppose he points his cathode-ray beam in the z-direction. The x-component of the beam deflection measures $\mathbf{B}_y$ and the y-component measures $\mathbf{B}_x$. Since he never observes any y-deflection he concludes $\mathbf{B}_x=0$. When he measures the x-deflection he finds that $\mathbf{B}_y$ oscillates sinusoidally with angular frequency ω and amplitude $\mathbf{B}_0$:

$$\mathbf{B}_y = \mathbf{B}_0 \sin\omega t.$$

Suppose that he next points his beam in the y-direction. Now the x-deflection measures $\mathbf{B}_z$ and the z-deflection measures $\mathbf{B}_x$. As before he finds $\mathbf{B}_x=0$. For $\mathbf{B}_z$ he finds an oscillating field with the same amplitude and the same frequency as for $\mathbf{B}_y$, but 90° out of phase with $\mathbf{B}_y$:

$$\mathbf{B}_z = \mathbf{B}_0 \cos\omega t.$$

He finds these same fields no matter where he measures on his disk. He can describe the field he measures as a uniform field $\mathbf{B}_0$ rotating with

angular velocity ω about his x-axis. If he were to compare notes with the laboratory observer, he would find his $\mathbf{B}_0$ the same as the laboratory man's B.

What about his "curl meter?" When he points it so that the axis of the loop is in the x-direction there is no flux through the loop and so the reading is always zero. Likewise his $\mathbf{B}_x=0$ always, so

$$\text{curl}_x\mathbf{E} = -\partial\mathbf{B}_x/\partial t = 0.$$

When he points the loop axis in the z-direction, however, there is a changing flux through the loop and he observes a reading which we can calculate as follows:

$$d\phi/dt = -\mathbf{B}_0 S\omega \sin\omega t, \qquad (11)$$

where S is the area of the loop. So

$$\oint \mathbf{E}\cdot d\mathbf{s} = \mathbf{B}_0 S\omega \sin\omega t. \qquad (12)$$

Then, from the definition of the curl,

$$\text{curl}_z\mathbf{E} = \mathbf{B}_0\omega \sin\omega t. \qquad (13)$$

This, however, is just what the disk observer measures for $-\partial\mathbf{B}_z/\partial t$. So

$$\text{curl}_z\mathbf{E} = -\partial\mathbf{B}_z/\partial t.$$

If the "curl meter" reading is calculated for the y-component we find that

$$\text{curl}_y\mathbf{E} = -\partial\mathbf{B}_y/\partial t.$$

No matter how an observer may be moving in a magnetic field, he always measures curl$\mathbf{E}=-\partial\mathbf{B}/\partial t$ in his own coordinate system.

Example No. 3

Now suppose that each observer is given some conducting wire and is told to try to arrange it so as to induce an emf in a closed circuit at rest in his own coordinate system. There is no way that the laboratory observer can arrange his conductor since $\partial\mathbf{B}/\partial t$ is everywhere zero. An equivalent statement, by virtue of the Maxwell equation, is to say that curl$\mathbf{E}=0$ everywhere. If curl$\mathbf{E}=0$ everywhere, then from Stokes' theorem $\oint\mathbf{E}\cdot d\mathbf{s}=0$ for any closed path.

The observer on the first disk, rotating about an axis perpendicular to its plane, has the same experience. $\partial\mathbf{B}/\partial t$ is zero everywhere and there is no way to induce an emf in a circuit at rest in this

coordinate system. If the disk itself is made of conducting material no eddy currents are induced in it.

Again things are different on the second disk, rotating about an axis in its plane. Suppose that the conductor is placed in a single loop around the rim of the disk, with a voltmeter connected in series with it. The voltmeter can be read by the laboratory observer and by the observer on the disk. The latter can calculate what the meter will read by invoking the Faraday law, since he knows $\partial B/\partial t$. He calculates

$$\text{emf} = -S(\partial B_z/\partial t) = B_0 S\omega \sin\omega t. \qquad (14)$$

He can also calculate the emf using Stokes' theorem, since he has measured curlE for every point on the disk. He calculates

$$\oint E \cdot ds = \int_S \text{curl}_z E \cdot dS = B_0 S\omega \sin\omega t, \qquad (15)$$

which is the same thing as before, of course, since $\text{curl} E = -\partial B/\partial t$.

Now suppose we calculate what the laboratory observer thinks the meter on the disk will read. To do this we can go straight to the Faraday law and calculate $d\phi/dt$. The laboratory observer says $d\phi/dt$ differs from zero because the circuit is rotating in a time-independent magnetic field, whereas the observer on the disk says $d\phi/dt$ differs from zero because his circuit, which is fixed, is experiencing a time-varying magnetic field. The laboratory man calculates $d\phi/dt$ through the rotating loop and gets

$$\text{emf} = BS\omega \sin\omega t, \qquad (16)$$

where his B is the same as the B_0 measured by the rotating observer.

The laboratory observer can also calculate the emf using Eq. (9):

$$\text{curl} E = -\partial B/\partial t + \text{curl}(v \times B).$$

He must use this equation, rather than Eq. (6), since the path around which he is calculating the emf is moving in his coordinate system. He says $\partial B/\partial t = 0$ everywhere, so that at every point on the disk

$$\text{curl} E = \text{curl}(v \times B). \qquad (17)$$

We then have, excluding terms whose curl is zero,

$$E = (v \times B). \qquad (18)$$
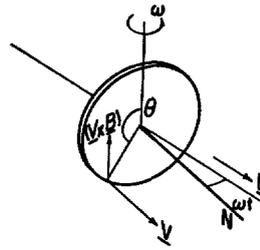

Fig. 3. Instantaneous values of v and $(v \times B)$ for the disk rotating about an axis in its plane. N is the instantaneous position of the normal to the disk.

Any terms with curl zero represent electrostatic fields given by the gradient of a scalar potential, about which we are not concerned here. The laboratory observer then says that

$$\oint E \cdot ds = \oint (v \times B) \cdot ds. \qquad (19)$$

At an arbitrary point on the rim of the disk $v = r \sin\theta\omega$ and $(v \times B) = Br\omega \sin\theta \sin\omega t$. So (see Fig. 3)

$$\oint (v \times B) \cdot ds = \int_0^{2\pi} Br\omega \sin\theta \sin\omega t \cos(90° - \theta) r d\theta$$

$$= \int_0^{2\pi} Br^2\omega \sin\omega t \sin^2\theta d\theta$$

$$= Br^2\pi\omega \sin\omega t$$

$$= SB\omega \sin\omega t,$$

which is the same result obtained by the Faraday law.

So the laboratory observer and the rotating disk observer agree on what the voltmeter reading should be but they disagree about the reason for the induced emf. The laboratory observer says the magnetic field is static but the Lorentz force $(v \times B)$ on the free charges in the moving conductor produces the emf. The disk observer says the conductor is at rest but it is in a time-dependent magnetic field.

In the foregoing discussion it should be noted that even if the disk is equipped with slip rings, so that the emf may be read on a meter at rest in the laboratory, it is really the electric field in the moving system which is being measured.

Example No. 4. The Unipolar Generator

Now suppose that we go to the first disk, rotating about an axis perpendicular to its plane, and
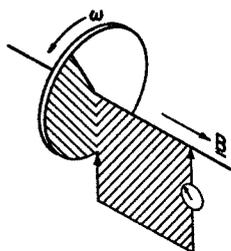
Fig. 4. The unipolar generator. The shaded portion represents the area through which the flux and the rate of change of flux are to be calculated.

place a conducting ring around the circumference so that we can make contact to it with a brush. Suppose we also place a conducting axle, equipped with a brush contact, through the disk. The axle and the ring are connected by a radial conducting wire, attached rigidly to the disk. The circuit is completed by a stationary wire with a voltmeter in series, as indicated in Fig. 4. The laboratory observer says that $\partial B/\partial t$ is everywhere zero, so that

$$\oint \mathbf{E} \cdot d\mathbf{s} = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{s}.$$

On the part of the circuit that is stationary in the laboratory $\mathbf{v} = 0$. On the rim and on the axle of the disk $(\mathbf{v} \times \mathbf{B})$ is everywhere perpendicular to the path of integration so that there is no contribution to the integral. Only along the radial conductor is there a contribution.

The laboratory observer calculates

$$\oint \mathbf{E} \cdot d\mathbf{s} = \int_0^R r\omega B dr = R^2 \omega B/2.$$

If he chose to use the Faraday law to calculate the emf, the result would be the same. In this case, however, we must specify carefully the surface through which the flux is to be calculated. It can be any surface bounded by the path of integration in the emf calculation. For convenience we may choose a surface lying in two planes: that of the axle and stationary parts of the circuit and that of the disk. The surface so defined is shaded in Fig. 4. The only part of this surface where the flux differs from zero is the part which lies on the disk. The rate of change of flux through this part of the surface is readily calculated.

If the observer on the disk calculates the emf, he also says $\partial B/\partial t$ is everywhere zero and he says the only place that $(\mathbf{v} \times \mathbf{B})$ differs from zero is in the portion of the circuit external to his disk.

He sees this part of the circuit rotating with respect to the disk with angular velocity ω. Again the only contribution to the emf is in the radial parts of the circuit, and he calculates

$$\oint \mathbf{E} \cdot d\mathbf{s} = (R^2 \omega B/2),$$

since the right radial wire is longer than the left radial wire by the radius R of the disk. The laboratory observer says there is an emf induced in the circuit because of the Lorentz force on the moving charges (free electrons) of the disk and the disk observer says there is an emf because of the Lorentz force on the moving charges of the portion of the circuit external to the disk, but they always agree on the voltmeter reading.

If the whole disk is made of conducting material the emf is calculated in exactly the same way. It makes no difference what integration path we choose from the axle to the rim as long as the path is at rest relative to the disk. It is essential that this part of the path be at rest relative to the conductor since the charges which are going to experience the force which results in the emf are at rest, on the average, in the conductor. The emf is independent of the path in the conductor since only the radial components of the path elements contribute to the $\int (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{s}$.

The emf may also be calculated by using the Faraday law. The only problem is how to choose the circuit through which the changing flux is to be calculated. Again we may choose any path in the moving conductor as long as it is at rest relative to the conductor. The problem is the same as with the wire from axle to rim, discussed in the foregoing.

The arguments made here are strictly valid only for large magnetic fields and small angular velocities, since centrifugal and Coriolis effects have been neglected. For practical laboratory purposes, however, this is no limitation at all. The ratio of Lorentz force to centripetal force on an electron in a disk rotating with angular velocity ω in a magnetic field $\mathbf{B}$ is

$$F_L/F_C = (e/m)(\mathbf{B}/\omega).$$

If the disk rotates at 1800 rpm in a field of one weber/m² (10 000 gauss) this ratio is 9.4×10^8. This complete separation of electrical and mechanical effects is a consequence of the large electronic charge to mass ratio.