

Pendulum Modeling and System ID

MAE 433 Spring 2012

Lab 6

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1 Overview

The next three labs address the control of an inverted pendulum balanced on a rotating beam. In this first lab, we derive a model for the system and perform some simple experiments to determine the parameter values. We also verify our models against experiments.

2 Deriving a dynamic model

Figure 1 shows a schematic of the rotary beam system.

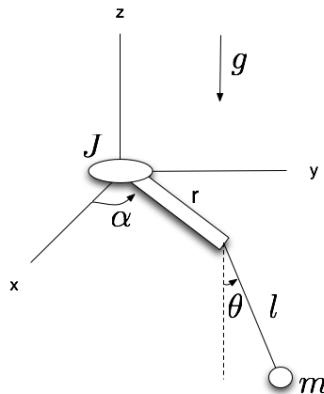


Figure 1: Diagram of rotary inverted pendulum.

2.1 Rotary pendulum dynamics

We can derive the equation of motion for the rotary pendulum system using Lagrange's equations, yielding

$$\begin{aligned}\ddot{\theta} + \frac{r}{l}\ddot{\alpha}\cos\theta - \dot{\alpha}^2\sin\theta\cos\theta - \frac{g}{l}\sin\theta &= 0 \\ (m+M)r^2\ddot{\alpha} + 2ml^2\dot{\alpha}\dot{\theta}\sin\theta\cos\theta + ml^2\ddot{\alpha}\sin^2\theta - mrl\dot{\theta}^2\sin\theta + mrl\ddot{\theta}\cos\theta &= T,\end{aligned}$$

where T is an applied torque. If we linearize about $\theta = \alpha = 0$, we get

$$\begin{aligned}\ddot{\alpha} &= -\frac{mgr}{J}\theta + \frac{T}{J} \\ \ddot{\theta} &= -\frac{(J+mr^2)g}{Jl}\theta + \frac{Tr}{Jl}.\end{aligned}\tag{1}$$

2.2 Motor dynamics

In the actual experiment, the input is not literally the force F or torque T , but is actually a voltage u applied to the motor. You may recall the following relations for a motor:

$$\tau = K_m i \quad e = K_m \omega.$$

The first equation is the ‘‘law of the motor’’ and relates the motor torque τ to the current i through the motor, where K_m is a constant. The second equation is the ‘‘law of the generator’’ which relates the shaft speed ω to the back EMF e induced by the motor.

If the shaft of a motor is connected so as to drive the rotary pendulum system, we obtain

$$T = K_1 u - K_2 \dot{\alpha}\tag{2}$$

where we have defined

$$\begin{aligned}K_1 &= K_m K_g / R \\ K_2 &= K_m^2 K_g^2 / R,\end{aligned}$$

R is the resistance of the motor, and K_g is the gear ratio from the motor to the shaft, which yields an additional relation

$$\dot{\alpha} = K_g \omega_{\text{motor}}.$$

2.3 Combined dynamics

Combining Equations (1) and (2), the equations of motion are

$$\begin{aligned}\ddot{\alpha} &= -\frac{mgr}{J}\theta + \frac{1}{J}(K_1 u - K_2 \dot{\alpha}) \\ \ddot{\theta} &= -\frac{(J+mr^2)g}{Jl}\theta + \frac{r}{Jl}(K_1 u - K_2 \dot{\alpha}).\end{aligned}\tag{3}$$

or, in state-space form,

$$\frac{d}{dt} \begin{bmatrix} \alpha \\ \theta \\ \dot{\alpha} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mgr}{J} & -\frac{K_2}{J} & 0 \\ 0 & -\frac{(J+mr^2)g}{Jl} & -\frac{rK_2}{Jl} & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \theta \\ \dot{\alpha} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{K_1}{J} \\ \frac{rK_1}{Jl} \end{bmatrix} u.\tag{4}$$

Using state vector $x = [\alpha \ \theta \ \dot{\alpha} \ \dot{\theta}]^T$ we can rewrite Equation (4) as

$$\dot{x} = Ax + Bu, \tag{5}$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mgr}{J} & -\frac{K_2}{J} & 0 \\ 0 & -\frac{(J+mr^2)g}{Jl} & -\frac{rK_2}{Jl} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{K_1}{J} \\ \frac{rK_1}{Jl} \end{bmatrix}.$$

3 System identification

To use the dynamic model described by Eqn. (4), we have to plug in values for all the parameters. Some of these are known constants, such as the gravitational acceleration g . Others are easy to measure directly, such as the pendulum length. But parameters such as the motor constants K_1 and K_2 are both unknown and difficult (or impossible) to measure, so we will determine them using carefully chosen experiments. The system identification procedure consists of three parts:

- *Pendulum dynamics:* Fix the rotary beam. Then determine a model for the pendulum alone, describing the downward hanging dynamics.
- *Rotary beam dynamics:* Remove the pendulum. Then determine a model for the rotary beam alone.
- *Combined:* Combine these models to obtain a model for the full system, describing the pendulum balanced in the up position.

Data from this week's lab will be used in future labs, so be sure to save your measured constants and responses.

3.1 Pendulum dynamics

The *linearized* equation of motion for a simple pendulum is given by

$$mL^2\ddot{\theta} + mgL\theta = 0.$$

If we assume a rigid body rather than a point mass on the end of a massless rod, we instead have

$$J\ddot{\theta} + mgl_c\theta = 0$$

where J is the moment of inertia of the pendulum, measured with respect about the pivot, and l_c is the distance from the pivot to the pendulum's center of mass.

Recalling that the second-order system

$$\ddot{y} + \omega_0^2 y = 0 \tag{6}$$

describes an oscillator with natural frequency ω_0 , we see that the simple pendulum has a natural frequency

$$\omega_{0,\text{simple}} = \sqrt{\frac{g}{L}},$$

whereas the rigid body pendulum has a natural frequency

$$\omega_{0,\text{rigid}} = \sqrt{\frac{mgl_c}{J}}.$$

We observe that if all the mass in the rigid body is concentrated at the end of the pendulum, then $l_c = L$, $J = mL^2$, and these reduce to the same thing.

If the pendulum has uniformly distributed mass, then

$$l_c = \frac{L}{2}$$

and

$$J = \frac{mL^2}{3},$$

so

$$\omega_0 = \sqrt{\frac{3g}{2L}},$$

and the “**effective length**” is

$$l = \frac{2L}{3}.$$

The effective length is the length of an equivalent simple pendulum. That is, a simple pendulum with length l will have the same dynamics as the rigid body pendulum with effective length l . We will determine the effective length l experimentally by measuring the natural frequency of the pendulum.

1. Download the file `lab6.zip` from Blackboard and unpack it. Move the files `nt_msvc.tmf` and `rotpend_template.mdl` to your working directory. Make a copy of `rotpend_template.mdl` and rename it.

We will need this template file for later labs, so you don't want to overwrite it.

2. Make sure that your rotary pendulum system is connected correctly. There should be a power cable connecting it to the UPM and two encoder cables connecting it to the Quanser board. *You should be able to identify the encoders on the rotary pendulum system. They look the same as the encoders on the DC motor test stands. The encoder for the rotary arm should be connected to channel 0. The encoder for the pendulum should be connected to channel 1.*
3. Open your model for the rotary pendulum system in Simulink and build it. Bring up a **Scope** and verify that the system behaves as you expect. Is the position absolute or relative? *Move the pendulum and rotary arm around. Make sure the signals corresponding to those variables change correctly. To determine if the position is absolute or not, try stopping the model and starting it again from another initial condition.*
4. Collect data to determine the natural frequency of the pendulum alone.
 - (a) Start your model with the pendulum hanging down.
 - (b) Hold the rotary beam steady.

- (c) Bring the pendulum up to some initial angle, then let go, allowing it to swing freely.
What initial angles make sense for this test? Think about problems that may occur in the limit of small and large angles.
- (d) Save the data to a `mat` file.
- (e) Turn off the model and remove the pendulum from the rotary arm. Measure its length and mass.
Leave the pendulum off, as we won't need it for the next part of the lab. Make sure to record the pendulum mass. We don't need its value now, but we will need it later.

5. Compute the effective length of the pendulum.

- (a) Load your data into Matlab and determine the period of oscillation from a plot θ .
In order to avoid any bias, take the average over many oscillation periods.
- (b) Compute the effective length from the equation

$$\omega_0 = \sqrt{\frac{g}{l}}.$$

- (c) Compare this to the theoretical value

$$l = \frac{2L}{3},$$

where L is the full length of the pendulum.

6. **Discuss your results with an AI before moving on to the next section.**

3.2 Rotary Beam dynamics

With the pendulum removed, the dynamics of the rotary beam are simply

$$J\ddot{\alpha} = \tau.$$

Plugging in from Eqn. (3), this is just

$$J\ddot{\alpha} + K_2\dot{\alpha} = K_1u.$$

In experiment, we can choose the input u and then measure the resulting values of α and $\dot{\alpha}$. Suppose we choose to apply proportional-derivative feedback such that

$$u = -(k_p\alpha + k_d\dot{\alpha}). \tag{7}$$

Then the governing equation becomes

$$J\ddot{\alpha} + (K_2 + K_1k_d)\dot{\alpha} + K_1k_p\alpha = 0 \tag{8}$$

This is just a standard second-order linear system, like a spring-mass-damper, where K_1k_p is the spring constant and $K_2 + K_1k_d$ is the damping coefficient. Recall that a second-order system of the form

$$\ddot{y} + 2\sigma\dot{y} + \omega_0^2y = 0 \tag{9}$$

has a solution

$$y(t) = e^{-\sigma t} \cos(\omega_d t),$$

where

$$\omega_d^2 = \omega_0^2 - \sigma^2.$$

Thus, solutions of Eqn. (8) will be damped sinusoids, and we can determine the various constants in Eqn. (8) by measuring the frequency ω_d and decay rate σ of the solutions.

P control

Eqn. (8) tells us that the rotary arm will behave like a damped oscillator for any choice of k_d . In this part of the lab, we will investigate whether or not pure proportional control ($k_d = 0$) is an appropriate way to modify the dynamics for system identification. Remember, we are trying to identify the decay rate and oscillation frequency of the system, which we can use to determine the model parameters.

1. Modify the Simulink model you downloaded so that it looks like the one shown in Fig. 2. Set the **Gain** block k_p to 8. Make sure the block **Gain1** is set to -1 , to get negative feedback. For this part of the lab you can set the **Constant** block `alpharef` to 0.

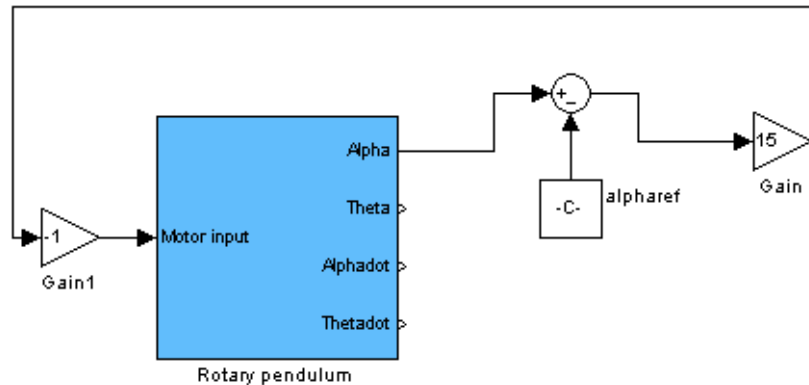


Figure 2: Simulink model for PD control.

2. Build the model and create two **Scopes**: one for α and one for `Motor_volts`.
3. Start the model and pull on the rotary arm to test its response. Does it behave as you expect? Try out different gains in the feedback loop, including small gains, large gains, and even (very small!) negative gains.
You don't have to stop the model between each test. The gain can be changed in real-time.
4. Determine if there is an appropriate gain value to use for system identification. **Discuss with an AI before proceeding.**
Hint: remember, we want to take experimental data from which we can identify a damping rate and an oscillation frequency. But the system identification procedure is only valid if the experimental data match the theoretical model. What physical effects cannot be captured by the model?

PD control

Part 1. Choosing appropriate gains

You should have seen in the previous part of the lab that pure proportional control did not yield a response suitable for system identification. (Did you figure out why?) In this part of the lab we will try using proportional-derivative control to modify the dynamics such that we get the desired response.

1. Modify your Simulink model from before to get the one shown in Fig. 3.

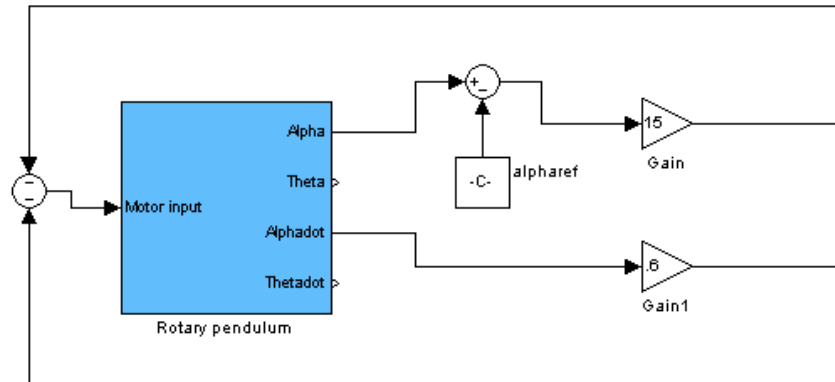


Figure 3: Simulink model for PD control.

2. Change the value of `alpharef` (α_r) to make the rotary beam move. Can you explain why this happens?
The reference angle is given in radians, not degrees.
3. Vary the gains k_p and k_d . Keep them within the ranges $0 \leq k_p \leq 15$ and $-0.6 \leq k_d \leq 0.6$. Choose values that give you a suitable response for system identification.
Remember, we are trying to identify a frequency and a damping rate from the experimental data. Can you clearly identify these values from your data?
4. Once you have found suitable gains, collect a step response of 20 degrees and **save the data for processing later**.

Part 2. Determining the moment of inertia

If we plot the step response collected above, we can determine the frequency ω_d and damping rate σ of this system. Comparing Eqn. (8) and (9), we see that

$$2\sigma = \frac{K_2 - K_1 k_d}{J}$$
$$\omega_0^2 = \frac{K_1 k_p}{J}.$$

If we knew the moment of inertia, J , we could calculate K_1 and K_2 using these equations. Unfortunately, we don't, so we will conduct another experiment to determine its value.

Suppose we add a mass to the end of the rotary arm. Then Eqn. (8) becomes

$$(J + m_0 r_0^2) \ddot{\alpha} + (K_2 + K_1 k_d) \dot{\alpha} + K_1 k_p \alpha = 0.$$

From this, we see that the frequency and damping rate of the system will change. (In the above expressions, simply replace J with $J + m_0 r_0^2$.) If we collect a step response for this new system and compare the new frequency ω'_d and damping rate σ' to the old ones (ω_d and σ), we can back out what the values of K_1 , K_2 , and J are.

1. Find the extra motor flywheel in your toolbox, weigh it, and secure it to the shaft of the rotary arm with a set screw.
Make sure the mass is well secured!
2. Measure the radius of the rotary arm r_0 , from its rotation axis to the point where the flywheel is mounted.
3. Using the same Simulink model from before, collect a step response for a reference angle of 20 degrees and **save the data for later processing**.
Make sure to use the same gains k_p and k_d from before.

Part 3. Calculating parameter values

In our Simulink model we added a reference `alpharef` that changes the control signal to

$$u = -\left[k_p(\alpha - \alpha_r) + c\dot{\alpha}\right]. \quad (10)$$

Thus the equation governing the above experiments is actually

$$J\ddot{\alpha} + (K_2 - K_1 k_d)\dot{\alpha} + K_1 k_p \alpha = K_1 k_p \alpha_r, \quad (11)$$

which has a response

$$\alpha(t) = \alpha_r \left[1 - e^{-\sigma t} \left(\cos(\omega_d t) + \frac{\sigma}{\omega_d} \sin(\omega_d t) \right) \right]. \quad (12)$$

We will use this equation to identify our unknown parameters. We now determine the unknown parameters of the systems, namely K_1 , K_2 and J .

1. Determine ω_d and σ .
 - (a) Plot the step response taken *without* the extra mass attached to the rotary arm.
 - (b) Estimate the frequency and guess a value for the damping rate.
You should be able to get a good estimate of the frequency, but don't worry about getting a good guess for the damping rate. Just guess!
 - (c) Plot the theoretical response given by Eqn. (12) on top of the experimental data.
 - (d) Adjust your values of ω_d and σ until the two curves match.
The two curves will likely not match perfectly. Which part of the data (beginning/end, large/small amplitudes) is more important to match?
 - (e) **Save this plot to turn in for your lab report, as well as your values of ω_d and σ .**
2. Determine ω'_d and σ' , using the same procedure as above.
 - (a) Compare the experimental step response (*with* the extra mass attached) to the theoretical response given by Eqn. (12).
Here, replace ω_d with ω'_d and σ with σ' when working with Eqn. (12).

- (b) Adjust your values of ω'_d and σ' until the two curves match.
- (c) **Save this plot to turn in for your lab report, as well as your values of ω'_d and σ' .**

3. Solve the overdetermined system

$$\begin{bmatrix} k_d & 1 & -2\sigma \\ k_p & 0 & -(\sigma^2 + \omega_d^2) \\ k_d & 1 & -2\sigma' \\ k_p & 0 & -(\sigma'^2 + \omega_d'^2) \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ J \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2m_0r_0^2\sigma' \\ m_0r_0^2(\sigma'^2 + \omega_d'^2) \end{bmatrix},$$

which we can write as $Mx = b$, using the command `x = M \ b`.

4. **Save your parameter values K_1 , K_2 , and J for later use.**
You will need these values for the next two labs, so don't lose track of them!

5. **Discuss with an AI before moving on.**

Combined dynamics and parameter validation

In the last part of this lab, we will combine the parameters determined above to get a model for the full rotary pendulum system. Then we will do an experiment to verify that our parameter values are reasonable.

1. Reconnect the pendulum to the rotary arm.
2. Using the same Simulink model from before, collect a step response, letting the pendulum swing for at least 4 or 5 periods. Save this data for later processing.
What value should you give for the step? For what values of θ and α are our linear models valid? What must be taken into account for large angles? For small angles?

3. Input the matrices A and B from Equation (5). Create a model of the open-loop system using the command

$$\text{sys} = \text{ss}(A, B, \text{eye}(4), 0).$$

4. Create a model for the closed-loop system using the command

$$\text{sys_closed} = \text{feedback}(\text{sys}, K, -1),$$

where the controller gain vector is given by $K = [k_p \ 0 \ k_d \ 0]$.

5. Simulate an initial condition response of the system using the command

$$[y, t] = \text{initial}(\text{sys_closed}, x_0, 0:dt:tend),$$

where $x_0 = [-\alpha_r \ 0 \ 0 \ 0]^T$.

6. Plot the simulated initial condition response on top of the experimental step response. Why is the initial condition response equivalent to a step response? **Discuss with an AI before proceeding.**

If your two plots don't agree well, think about whether or not your initial condition is valid. They should at least agree for the first oscillation or two. If your initial condition is valid and your plots still don't agree, then you will have to repeat your system identification procedure.

7. Save your plot and print it out for your lab report.

4 Deliverables

- Plots of curve fits used to find ω_d , σ , ω'_d , and σ' , along with their values
- Comparison of experimental step response to simulated initial condition response