

# On Eddy Currents in a Rotating Disk

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A DEVICE which often occurs in electric machines and instruments consists of a relatively thin conducting disk rotating between the pole pieces of a permanent magnet or electromagnet. The author has received inquiries as to the method of calculating the paths of the eddy currents and the torque in such cases. The following rather simple method, which is quite accurate for a permanent magnet, seems not to be described in the literature. It assumes that the disk is so thin that the skin effect can be neglected. This is true for all frequencies that can be produced mechanically. To facilitate calculation in the special case of circular poles it is also as-

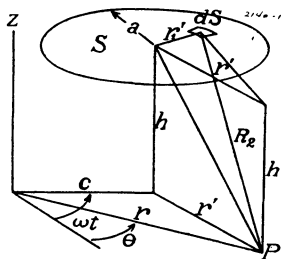


Figure 1. Geometrical relations for derivation of formulas for stream function

sumed that  $2\pi\omega ab\gamma = \epsilon a$  is much less than  $\omega a$  where  $\omega$  is the angular frequency of rotation in radians per second,  $a$  the pole-piece radius,  $b$  the disk thickness, and  $\gamma$  the electric conductivity, all in centimeter-gram-second electromagnetic units. This produces a fractional error of less than  $\epsilon a$  in the eddy current densities and of less than  $(\epsilon a)^2$  in the torque. In the case of the electromagnet the situation is complicated by the presence of the permeable pole pieces in the magnetic field of the eddy currents. This may send a large demagnetizing flux through the electromagnet. An approximate solution for this case will be considered.

## Maxwell's Formula

This calculation starts from a formula given by Maxwell in 1873,<sup>1</sup> but apparently little known to engineers. To apply it one

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should know its derivation which, as given by Maxwell, is difficult for modern students to follow. A simplified proof which brings out the points essential for our problem is given below.

The object is to calculate the magnetic induction  $\mathbf{B}$  produced by the eddy currents of density  $\mathbf{i}$  induced in a thin plane sheet of thickness  $b$ , unit permeability and conductivity  $\gamma$  lying in the  $xy$  plane by a fluctuating magnetic field of induction  $\mathbf{B}'$ . Evidently the only components of  $\mathbf{i}$  effective in producing magnetic effects parallel its surface. Let the eddy currents be confined to a finite region of the sheet which may or may not extend to infinity, and let us define the stream function  $U(x, y)$  at any point in the sheet to be the current flowing through any cross section of the sheet extending from  $P$  to its edge. The line integral of  $\mathbf{B}$  or  $\mathbf{H}$  over the closed path that bounds this section equals  $4\pi U$ . From symmetry the contribution from the upper and lower halves of the path is the same so we may write

$$4\pi U = \oint \mathbf{B} \cdot d\mathbf{s} = \pm 2 \int_x^\infty B_x dx = \pm 2 \int_y^\infty B_y dy \quad (1)$$

where the choice of sign depends on the side of the sheet chosen for the integration. Differentiating this equation gives

$$b i_x = \frac{\partial U}{\partial y} = \pm \frac{B_y}{2\pi} \quad b i_y = -\frac{\partial U}{\partial x} = \mp \frac{B_x}{2\pi} \quad (2)$$

These equations connect the eddy current density with the tangential components

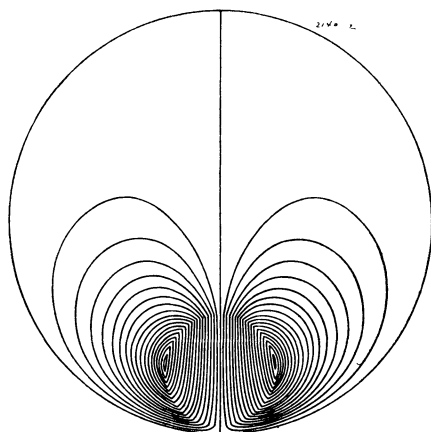


Figure 2. Lines of flow of eddy currents induced in rotating disk by single circular magnet pole

of the magnetic induction  $\mathbf{B}$  produced by  $\mathbf{i}$  at the surface of the sheet.

The eddy currents are generated not only by the changes in the magnetic induction  $\mathbf{B}'$  of the external field, but also by the changes of the magnetic induction  $\mathbf{B}$  of eddy currents elsewhere in the sheet. One of Maxwell's equations combined with Ohm's law gives the induced current to be

$$\nabla \times \mathbf{E} = \nabla \times \frac{\mathbf{i}}{\gamma} = -\frac{\partial}{\partial t} (\mathbf{B}' + \mathbf{B}) \quad (3)$$

Writing out the  $z$  component of this equation and using equation 2 give

$$\frac{1}{\gamma} \left( \frac{\partial i_y}{\partial x} - \frac{\partial i_x}{\partial y} \right) = \mp \frac{1}{2\pi b \gamma} \left( \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} \right) = -\frac{\partial}{\partial t} (B_z' + B_z) \quad (4)$$

Another of Maxwell's equations states that

$$\nabla \cdot \mathbf{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \quad (5)$$

Combining equations 4 and 5 gives

$$\pm \frac{\partial (B_z' + B_z)}{\partial t} = \frac{1}{2\pi b \gamma} \frac{\partial B_z}{\partial z} \quad (6)$$

When  $\partial B_z' / \partial t$  is known, this equation gives the boundary condition on  $B_z$  in the plane of the sheet. This, combined with the equations  $\nabla \times \mathbf{B} = 0$  and  $\nabla \cdot \mathbf{B} = 0$  which hold outside the sheet, and the fact that  $\mathbf{B}$  vanishes at infinity serves to determine  $\mathbf{B}$  everywhere. By equations 1 and 2 the current density and stream function anywhere in the sheet can be found.

The explicit expression for  $\mathbf{B}$  in terms

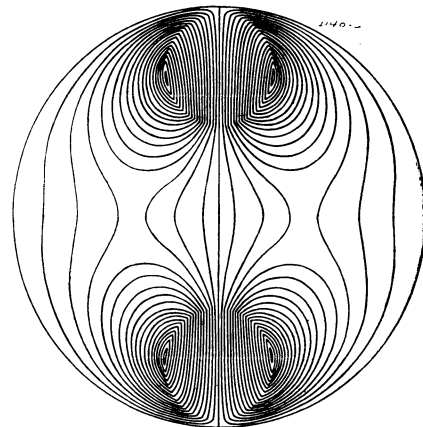


Figure 3. Lines of flow of eddy currents induced in rotating disk by two circular magnet poles

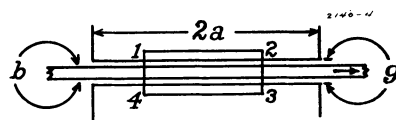


Figure 4. Geometrical relations for calculation of demagnetizing flux

of  $B'$  which was given by Maxwell<sup>1</sup> can be obtained as follows. The right side of equation 6 is finite at all times which means that if  $\Delta t \rightarrow 0$  then  $\Delta(B_2' + B_2) \rightarrow 0$ . Thus an abrupt change in  $B'$  instantaneously induces eddy currents such as will maintain  $B' + B$  unchanged in the sheet. Therefore, for a specified change in  $B'$  the initial value of  $B$  is known, and, if no further changes in  $B'$  occur its subsequent values as the eddy currents decay are found by putting  $\partial B'/\partial t = 0$  in equation 4 and solving. A second abrupt change in  $B'$  produces a second set of eddy currents, and so forth. At any instant the actual field of the eddy currents is a superposition of these. As the magnitudes of the discontinuous changes in the external field become smaller, and the intervals between them shorter, we approach as a limit a continuously changing magnetic field.

Suppose that the sources of the inducing field lie above the  $xy$  plane where  $z > 0$ . At  $t = 0$  the source changes abruptly the induction being  $B_1' = F_1(x, y, z)$  when  $-\infty < t < 0$  and  $B_2' = F_2(x, y, z)$  when  $0 < t < \infty$ . As just shown the eddy currents generated at  $t = 0$  initially keep the field on the negative side of an infinite sheet unchanged. When  $z < 0$  we have therefore

$$B_{t=0} = B_1' - B_2' = F_1(x, y, z) - F_2(x, y, z) \quad (7)$$

Since  $B_2'$  is not a function of  $t$ , equation 6 reduces to

$$\pm \frac{\partial B_2}{\partial t} = \frac{1}{2\pi b \gamma} \frac{\partial B_2}{\partial z} \quad (8)$$

These equations, 7 and 8, are satisfied by

$$B = F_1\left(x, y, z \pm \frac{t}{2\pi b \gamma}\right) - F_2\left(x, y, z \pm \frac{t}{2\pi b \gamma}\right) \quad (9)$$

Because the eddy currents must die out, and their magnetic field must be symmetrical about the sheet, we take the plus sign when  $z$  is positive and the negative sign when  $z$  is negative. Thus equation 9 shows that, in addition to  $B_2'$  which would exist if no sheet were present, there is a decaying field due to eddy currents which appears, from either side of the sheet, to be caused by a pair of images receding with uniform velocity  $1/(2\pi b \gamma)$ . Suppose our inducing field has the form

$$B' = F(t, x, y, z) \quad (10)$$

The change in this field in an infinitesimal time interval  $d\tau$  is given by

$$\frac{\partial B'}{\partial t} d\tau = \frac{\partial}{\partial t} F(t, x, y, z) d\tau \quad (11)$$

The initial field of the eddy currents formed in that interval must be equal and opposite to this and must die out as if their source moved away with a uniform speed  $1/(2\pi b \gamma)$ . Thus the eddy currents at a time  $t$  due to a change in the interval  $d\tau$  at a time  $\tau$  before  $t$  is given by

$$dB = -\frac{\partial}{\partial t} F\left(t - \tau, x, y, z \pm \frac{\tau}{2\pi b \gamma}\right) d\tau \quad (12)$$

This is Maxwell's formula. It has many applications.<sup>2</sup> When the field is produced by moving permanent magnets, it is convenient to express  $U$  in terms of the scalar magnetic potential  $\Omega$ . Since we have unit permeability we may write

$$U = \frac{1}{2\pi} \int_r^\infty B_r dr = \frac{-1}{2\pi} \int_r^\infty \frac{\partial \Omega}{\partial r} dr = \frac{\Omega}{2\pi} \quad (13)$$

### Application to Magnet Moving in a Circle

We now take the case of a magnetic field produced by a long right circular cylinder of radius  $a$ , uniformly and permanently magnetized parallel to its axis, so as to give a total flux  $\Phi$ . The magnetic pole density in the face is therefore  $\Phi/(2\pi a)^2$ . This magnet moves in a circle with a uniform angular velocity  $\omega$  its axis

unprimed or primed according as they refer to the axis of rotation or to the pole-piece axis. The scalar magnetic potential  $\Omega'$  of its lower face, of area  $S$ , at the point  $P$  is seen from Figure 1 to be

$$\Omega' = \frac{\Phi}{(2\pi a)^2} \int_S \frac{dS}{R_2} = \frac{\Phi}{(2\pi a)^2} \times \int_0^a \int_0^{2\pi} \frac{r_1' dr_1' d\theta'}{\sqrt{h^2 + r_1'^2 + r'^2 - 2r_1' r' \cos \theta'}}$$

where  $r'^2 = r^2 + c^2 - 2rc \cos(\omega t + \theta)$ . This combined with equations 12 and 13 give the stream function to be

$$U = \frac{\Phi}{(2\pi)^3 a^2} \int_r^\infty \int_0^{2\pi} \int_0^\infty \frac{\partial}{\partial t} \times \left( \frac{r_1' dr_1' d\theta' d\tau}{\sqrt{h_\tau^2 + r_1'^2 + R_\tau'^2 - 2r_1' R_\tau' \cos \theta'}} \right) \quad (14)$$

where  $R_\tau'^2 = r^2 + c^2 - 2rc \cos(\omega(t - \tau) + \theta)$  and  $h_\tau = h + \tau/(2\pi b \gamma) = h + u$ . Let us now bring the pole piece down close to the plate so that  $h_\tau = \tau/(2\pi b \gamma)$ , and bring up a similar pole of opposite sign from the other side, so that the eddy current density is doubled. We now carry out the differentiation with respect to  $t$  and set  $t = 0$  so that the  $\theta = 0$  line bisects the pole piece when  $t = 0$ . The integral then becomes

$$U = \frac{r c \Phi \epsilon}{\pi (2\pi a)^2} \int_0^a \int_0^{2\pi} \int_0^\infty \frac{\sin(\theta + \epsilon u) (R_u' - r_1' \cos \theta') r_1' dr_1' d\theta' du}{R_u' (u^2 + r_1'^2 + R_u'^2 - 2R_u' r_1' \cos \theta')^{3/2}} \quad (15)$$

being  $c$  centimeters from the  $z$  axis, and its lower end  $h$  centimeters above the  $z = 0$  plane in which lies an infinite plane sheet of thickness  $b$  and conductivity  $\gamma$ . Its upper end is too remote for consideration. Polar co-ordinates will be written

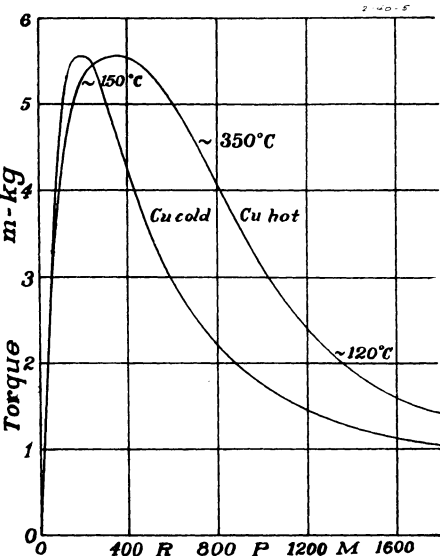


Figure 5. Curves showing torque versus speed for large disk rotating between the four rectangular pole pairs of an electromagnet, measured by Lentz

where  $\epsilon = 2\pi \omega b \gamma$  and  $R_u'^2 = r^2 + c^2 - 2rc \cos(\theta + \epsilon u)$ . For 3,000 rpm with a copper sheet 0.25 millimeter thick  $\epsilon \approx 0.01$  so that  $u^2$  in the denominator has reached the value 100 when  $\epsilon u$  reaches 0.1. In calculating such a quantity as the torque where the current density is integrated over the pole piece, the neglect of  $\epsilon$  produces a fractional error less than  $(\epsilon/a)^2$ , so that the result should be good to one per cent for a sheet one millimeter thick. We may therefore drop the  $\epsilon$  terms so that  $R_u'$  becomes the  $r'$  in Figure 1 and integrate with respect to  $u$  giving

$$U = \frac{K \sin \theta}{r'} \times \int_0^a \int_0^{2\pi} \frac{(r' - r_1' \cos \theta') r_1' dr_1' d\theta'}{r'^2 + r_1'^2 - 2r' r_1' \cos \theta'} \quad (16)$$

where we have written  $K$  for the coefficient of the integral in equation 15. The integral with respect to  $\theta'$ , from Dwight's table of integrals 860.2, is zero when  $r' < r_1'$  and  $\pi/r'$  when  $r' > r_1'$ . Thus the upper limit for the  $r_1'$  integration is  $a$  when  $r' > a$  and  $r'$  when  $r' < a$ , which gives

$$r' > a \quad U = \frac{\omega r c b \gamma \Phi \sin \theta}{2\pi r'^2} \quad (17)$$

$$r' < a \quad U = \frac{\omega r c b \gamma \Phi \sin \theta}{2\pi a^2} \quad (18)$$

The next question is how to restrict the eddy currents to the interior of the disk bounded by the circle  $r=A$ . We observe that if we use equation 15 to calculate  $U$  for a second magnet also carrying a flux  $\Phi$  but with circular pole pieces of radius  $a''=Aa/c$  centered at  $c''=A^2/c$ , so that  $R''^2=r^2+(A^2/c)^2-2r(A^2/c)\cos(\theta+\epsilon u)$ , and change the variables of integration from  $r_1'$  to  $r_1'A/c$ , and from  $u$  to  $Au/c$ , then the resultant expression is identical with equation 15, except that we have  $cR_u''/A$  instead of  $R_u'$  and  $A\epsilon/c$  instead of  $\epsilon$ . But when  $r=A$  we see that  $cR_u''/A=R_u'$ , so that both magnets, one outside and one inside the circle  $r=A$ , give the same  $U$  on this circle. Furthermore by taking the air gap in each magnet small, the fluxes are confined to the areas under the pole pieces, so that neither induces directly eddy currents on the other side of the circle  $r=A$ . It is evident that if the fluxes from the two magnets cut the sheet in opposite directions, then  $U=0$  when  $r=A$  and the currents induced by the inner pole are kept inside the circle. This is exactly the boundary condition for a disk of radius  $A$ , except that the calculated system includes the currents induced in the region  $r<A$  by the magnetic field of the eddy currents in the region  $r>A$ , which does not exist in the case of the disk. This field is proportional to  $\Phi$  which is, by hypothesis, small compared with  $\Phi$ , and in addition the source is further away, so that the fractional error in  $U$  will be less than  $\epsilon$ . We should note also from the symmetry that the radial component of these secondary currents is opposite in sign on the two sides of the  $\theta=0$  line, so that their effect cancels out completely in calculating the torque which therefore should be accurate to terms in  $\epsilon^2$ . The contribution to  $U$  from the outer magnet is found by putting  $c^2R''^2/A^2$  for  $R^2$  in equation 17. Adding this to equations 17 and 18, we obtain for the stream function of the eddy currents in the disk

$$R>a \quad U = \frac{\omega r c b \gamma \Phi \sin \theta}{2\pi} \left( \frac{1}{r^2+c^2-2rc \cos \theta} - \frac{A^2}{c^2r^2+A^4-2rcA^2 \cos \theta} \right) \quad (19)$$

$$R<a \quad U = \frac{\omega r c b \gamma \Phi \sin \theta}{2\pi a^2} \left( 1 - \frac{A^2 a^2}{c^2r^2+A^4-2rcA^2 \cos \theta} \right) \quad (20)$$

The torque may be calculated by integrating the product of the radial component of the current by the magnetic induction and by the lever arm and integrating over the area  $S$  of the pole piece.

Thus, using equation 2, we have

$$T = \int_S \frac{r b i_r \Phi}{\pi a^2} dS = \frac{\Phi}{\pi a^2} \int_{c-a}^{c+a} \int_{-\theta_1}^{+\theta_1} r \frac{\partial U}{\partial r} r dr d\theta$$

where  $\theta_1$  and  $r$  are connected by the relation  $r^2+c^2-2rc \cos \theta_1=a^2$ . Substituting for  $U$  from equation 20 and integrating with respect to  $\theta$  give

$$T = \frac{\omega c b \gamma \Phi^2}{\pi^2 a^4} \times \int_{c-a}^{c+a} \left( r^2 \sin \theta_1 - \frac{a^2 A^2 r^2 \sin \theta_1}{c^2 r^2 + A^4 - 2A^2 r c \cos \theta_1} \right) dr \quad (21)$$

The integration is simplified by taking a new variable  $u$  so that  $4acu^2=r^2-(c-a)^2$  which gives the limits 0 and 1. Thus we obtain, writing out  $\epsilon$ ,

$$T = \frac{\omega b \gamma \Phi^2 c^2}{2\pi a^2} \left( 1 - \frac{A^2 a^2}{(A^2 - c^2)^2} \right) = \omega \gamma \Phi^2 D_1 \quad (22)$$

This formula gives the torque in dyne centimeters when  $\omega$  is in radians per second,  $\Phi$  in maxwells,  $a$ ,  $b$ ,  $c$ , and  $A$  in centimeters and  $\gamma$  in electromagnetic units. If we are given the volume resistivity  $\rho$  of the disk in ohm-centimeters  $\gamma = 10^{-9}/\rho$ .

If the magnet is fixed, and the disk rotates, the arrangement described exerts an undesired force on the disk axis which may be avoided by using two identical magnets on opposite sides of the axis and equidistant from it. This approximately doubles the torque given by equation 22. The additional torque from the eddy currents of one magnet flowing under the poles of the other may be found by an integral similar to equation 21 which is

$$T' = \frac{\omega b \gamma \Phi^2 c}{\pi^2 a^2} \int_{c-a}^{c+a} \left( \frac{r^2 \sin \theta_1}{r^2+c^2+2rc \cos \theta_1} - \frac{r^2 A^2 \sin \theta_1}{r^2 c^2 + A^4 + A^2 r c \cos \theta_1} \right) dr \quad (23)$$

Integrating by the same substitutions as equation 21, adding to equation 22 and multiplying by two give

$$T = \frac{\omega b \gamma \Phi^2 c^2}{\pi a^2} \left( \frac{4c^2+a^2}{4c^2} - \frac{2a^2 A^2 (A^4+c^4)}{(A^4-c^4)^2} \right) = \omega \gamma \Phi^2 D_2' \quad (24)$$

This holds when the two magnet fields are antiparallel. If we subtract the integral of equation 23 from equation 22 and multiply by two we get

$$T = \frac{\omega b \gamma \Phi^2 c^2}{\pi a^2} \left( \frac{4c^2-a^2}{4c^2} - \frac{4a^2 c^2 A^4}{(A^4-c^2)^2} \right) = \omega \gamma \Phi^2 D_2'' \quad (25)$$

This holds when the two magnetic fields are parallel. The arrangement of equation 24 gives more torque than that of equation 25. The eddy-current flow lines corresponding to constant values of  $U$  as

calculated from equations 19 and 20 applied to the cases of equations 22 and 24 are shown in Figures 2 and 3 where  $a = \sqrt{7}$  cm,  $c=7$  cm,  $A=10$  cm and  $\omega b \gamma \Phi / (2\pi) = 3.5$ . The value  $U$  on the outer boundary is zero and changes by steps of one in Figure 2 and steps of two in Figure 3.

## Demagnetizing Effects

So far the magnet pole pieces have been assumed to be so hard that they do not short-circuit the flux of the eddy currents. This is not true for the permeable pole pieces of an electromagnet, whose effect may be calculated approximately by observing that the current  $2U$  is enclosed by the rectangular path 1-2-3-4-1 in Figure 4, which lies in the upper and lower pole pieces except where it cuts across the disk and gap normally at  $r=r_1$  and  $\theta = \pm \theta_1$ . If the reluctance of this circuit lies entirely in the air gaps, each of length  $g$ , then the magnetic flux density  $B_e$  due to the eddy currents alone at  $r_1, \pm \theta_1$  is  $4\pi U/g$ . Substituting for  $U$  from equation 20 and writing as before  $\epsilon = 2\pi \omega b$  gives

$$B_e = \frac{c \epsilon r_1 \Phi \sin \theta_1}{\pi a^2 g} \left( 1 - \frac{A^2 a^2}{c^2 r_1^2 + A^4 - 2r_1 c A^2 \cos \theta_1} \right) \quad (26)$$

This shows that when  $b$  and  $g$  are comparable in size  $B_e$  cannot be neglected compared with the original flux density  $\Phi/(\pi a^2)$ . The  $\sin \theta_1$  term shows that the radial component of the eddy currents induced by  $B_e$  have opposite signs under the two halves of the pole piece, so that they contribute nothing directly to the torque, but on the other hand they form closed circuits about the central portion and so produce a demagnetizing magnetomotive force in the electromagnet. The

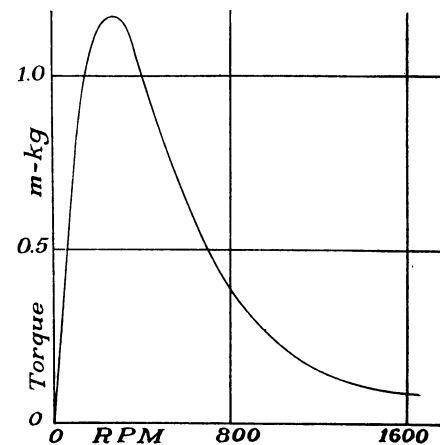


Figure 6. Curves showing torque versus speed for a large disk rotating between the single pair of circular pole pieces of an electromagnet as given by equations 22, 27, and 28

stream function  $U_e$  of these eddy currents is calculated as  $U$  is, but to simplify matters we carry out the operations from equations 14 to 16 for a single element of the pole face, along with its image element outside the circle  $r=A$ . We then give each element the strength indicated in equation 26 and set up a definite integral for  $U$  over the pole-piece area. This method is less exact than setting up equation 14 for the whole face, because it ignores that part of the flux threading  $dS$  from the current induced by  $B_e$  outside this area, which is of the order  $\epsilon B_e dS$ . The eddy currents  $U_e$  are evidently equivalent to a magnetic shell of variable strength  $U_e$  in the sheet and to get  $\mathcal{F}_e$  the demagnetizing magnetomotive force we must find the equivalent uniform shell. Thus we have

$$\mathcal{F}_e = \frac{1}{\pi a^2} \int_s U_e dS$$

where  $S$  is the area of the pole face. We now have a complicated quadruple integral involving the variables  $r'$ ,  $R'$ ,  $\theta'$  and  $\theta_1'$  whose evaluation can be simplified somewhat by integrating in the proper order. The result is

$$\mathcal{F}_e = \frac{\omega^2 b^2 \gamma^2 c^2}{4g} \Phi \left( 1 - \frac{2a^2 A^2 (A^2 + c^2)}{c^2 (A^2 - c^2)^2} + \frac{2A^4}{c^4} \log_e \frac{(A^2 - c^2)^2}{(A^2 - c^2)^2 - c^2 a^2} \right) = \beta^2 \gamma^2 \omega^2 \Phi \quad (27)$$

If the flux penetrating the sheet at rest is  $\Phi_0$ , then when in motion we have, if  $\mathcal{R}$  is the reluctance of the electromagnet,  $\Phi = \Phi_0 - \beta^2 \gamma^2 \omega^2 \Phi / \mathcal{R}$ , so that

$$\Phi = \frac{\mathcal{R} \Phi_0}{\mathcal{R} + \beta^2 \gamma^2 \omega^2} \quad (28)$$

The expressions for the torque now become

$$T = \frac{\omega \gamma \mathcal{R}^2 \Phi_0^2 D}{(\mathcal{R} + \beta^2 \gamma^2 \omega^2)^2} \quad (29)$$

where  $D$  has the values given in equations 22, 24, or 25, according to the pole arrangement. There is now a definite speed for maximum torque which is found by setting  $\partial T / \partial \omega = 0$  to be

$$\omega_m = \frac{1}{\beta \gamma} \sqrt{\frac{\mathcal{R}}{3}} \quad (30)$$

Putting this in equation 29 gives

$$T_m = \frac{3 \sqrt{3} \mathcal{R} \Phi_0^2 D}{16 \beta} \quad (31)$$

This is independent of the conductivity which is surprising, although there is some evidence for it in Lentz's experimental

curves shown in Figure 5 which give hot and cold copper disks the same  $T_m$  for different  $\omega_m$ .

These calculations of demagnetizing effects have been worked out for a single pole. For an even number of poles with alternating signs, we have seen that the torque per pole is increased, but the demagnetizing forces are also increased so that the torque obtained by multiplying equation 29 by the number of poles will probably not be far wrong. The speed for maximum torque given by equation 30 will certainly be decreased, perhaps considerably, because of the increase in  $\beta$ .

The only formula we can find for this torque is one derived by Rüdénberg.<sup>3</sup> This formula is written as a double infinite series and is derived by considering a thin conducting strip bounded by straight lines which moves lengthwise in the narrow gap between magnetic poles with rectangular faces. The fields of adjacent poles are antiparallel, so that the inducing fields can be expanded in a double series of odd harmonics. This formula was checked qualitatively by Zimmermann,<sup>4</sup> but could not be verified quantitatively, as the theoretical and experimental boundary conditions did not agree. Lentz found only those terms involving the lengthwise harmonics were of importance and dropped the rest. His experimental brake had the center of the disk removed to simulate a ring whose width roughly equaled that of the postulated strip. His four poles were so far apart that their action was nearly independent. We have redrawn in Figure 5, his experimental curves giving the torque in meter kilograms against angular velocity in revolutions per second. The ring had inner and outer radii of 5 centimeters and 25 centimeters and was 0.4 centimeter thick. The air space was 1.2 centimeters, and the centers of the rectangular pole pieces were 20.75 centimeters from the rotation axis and were 6 centimeters (radial) by 8 centimeters (tangential). The inducing field was 2,150 gauss at rest. The figures on the hot copper curve show the estimated stable mean temperature for that speed.

A direct quantitative comparison of our formula with Lentz's data is difficult, because he used rectangular poles, his air gap was so large as to spread the inducing field over an unknown area, the center of his disk was cut away, and we do not know where his flux density was measured. Although our formulas are inaccurate for such large dimensions at the

high speeds, it is interesting to see what results they give for a comparable case. Let us take  $b=0.4$  cm,  $A=25$  cm,  $a=4$  cm,  $c=21$  cm,  $g=0.6$  cm,  $B=2,000$  gauss and assume the reluctance entirely in the air gap. In equation 22,  $D_1=1.23$ , in equation 27,  $\beta=3.85$  and in equation 28  $\mathcal{R}=0.012$ . The angular velocity for maximum torque for copper ( $\gamma=1/1,700$ ) is given by equation 30 to be 27.9 radians per second or 267 rpm.  $T_m$  is  $1.15 \times 10^3$  dyne cm or 1.17 kilogram-meters for this single pole and roughly four times this for four poles. Expressing  $T'$  in kilogram-meters and  $\omega'$  in rpm, equation 29 becomes

$$T' = \frac{0.00785 \omega'^2}{(1 + 0.000047 \omega'^2)^2} \text{ kg-m}$$

This formula is plotted in Figure 6. A comparison of Figure 5 with Figure 6 indicates that our formula gives too rapid a falling off in torque at high speeds. It should be pointed out that other conditions, such as the degree of saturation of the iron in the magnet will upset the assumed relation between magnetomotive force and  $\Phi$  and may modify equations 28, 29, 30, and 31 considerably.

The methods given in this paper may be extended to any number of poles by the method used for two and to other than circular faces. Several such calculations have been carried out, but it is doubtful if the additional theoretical accuracy justifies publishing them. The difference between the ideal boundary conditions used here and those found in apparatus is such that we recommend that the torque for one pole be calculated by equation 22 for permanent magnets or by equation 29 for electromagnets, and the result multiplied by the number of poles to give the total torque. In power apparatus the heating of the disk will change its resistivity and may cause it to expand and buckle and otherwise upset the calculations.

## References

1. ELECTRICITY AND MAGNETISM, Maxwell. Oxford University Press, 1892, volume 2, page 297.
2. STATIC AND DYNAMIC ELECTRICITY, Smythe. McGraw-Hill Book Company, New York, N. Y., 1939. Chapter XI.
3. SAMMLUNG ELEKTROTECHNISCHE VORTRÄGE. Verlag Enke, Stuttgart 1907. Bd 10. page 269 and following.
4. *Elektrotechnik und Maschinenbau*, Bd 40, 1922. page 11.
5. *Elektrotechnik und Maschinenbau*, Bd 52, 1934, pages 99-102.